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Ergodicities and exponential ergodicities of branching processes with immigration

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1. Motivation

A random variable X is said to be self-decomposable if for every $t \geq 0$ there is random variable Y_t independent of X such that $e^{-t}X + Y_t \overset{d}{=} X$.

Also say the distribution μ of X is self-decomposable, i.e., for $t \geq 0$,

$$(\mu Q_t) * \gamma_t = \mu \qquad (\gamma_t = \mathscr{L}(Y_t),\ Q_t(x, \cdot) = \delta_{e^{-t}x}). \tag{1}$$

Representation (Sato–Yamazato '84)**:** *There is a unique i.d. probability* ν *on* $\mathbb{R}^d$ *so that* $\int_{\mathbb{R}^d} \log(1+|x|)\nu(ds) < \infty$ *and* (" ˆ " means characteristic function)

$$\hat{\mu}(\lambda) = \exp\left\{\int_0^\infty \log \hat{\nu}(e^{-t}\lambda)dt\right\}, \qquad \lambda \in \mathbb{R}^d. \tag{2}$$

Remarks: (i) Laws $\lim_{n\to\infty} \mathscr{L}(\sum_{i=1}^n \frac{X_i - a_n}{b_n})$; (ii) Equilibria of OU-type processes.

Generalization (Van Harn et al. '82)**:** $(Q_t)_{t\geq 0}$ = branching process; μ = equilibrium of an immigration model. **Problem:** Ergodicities and exponential ergodicities.

The literature

Sato–Yamazato ('84): necessary and sufficient condition for ergodicity of OU-type processes.

Schilling–Wang ('12), Wang ('12): ergodicity and exponential ergodicity of OU-type processes in total variation distance.

Wang–Wang ('13): OU-type processes in infinite-dimensional state spaces (Banach spaces).

Pinsky ('72): necessary and sufficient condition for ergodicity of continuous-state branching processes with immigration (CBI-processes).

Jin–Kremer–Rüdiger ('18+), Friesen–Jin–Rüdiger ('19+): ergodicity and exponential ergodicity of affine processes in weak convergence and in Wasserstein distances.

Stannat ('03a, '03b): Dawson–Watanabe superprocesses with immigration, compact metric spaces, Feller type assumptions, Wasserstein and total variation distances.

Friesen ('19+): extensions to Lusin spaces, Borel right setting, Wasserstein distance.

An example of application

A stable CIR-model is a continuous-state branching process with immigration $\{X_t : t \geq 0\}$, constructed by the strong solution to (Fu–L' 10):

$$dX_t = \sqrt[\alpha]{\alpha c X_{t-}} dZ_t - bX_{t-}dt + adt, \qquad (3)$$

where $\{Z_t\}$ is a Brownian motion ($\alpha = 2$) or a one-sided α-table process ($1 < \alpha < 2$).

- Let $\{X_0, X_1, \cdots, X_n\}$ be low frequency observations of the stable CIR-model.
- We have the consistent (conditional least squares) estimators (L–Ma '15):

$$-\log \frac{\sum_{k=1}^n X_{k-1} \sum_{k=1}^n X_k - n \sum_{k=1}^n X_{k-1} X_k}{\big(\sum_{k=1}^n X_{k-1}\big)^2 - n \sum_{k=1}^n X_{k-1}^2} =: \hat{b}_n \xrightarrow{\text{p}} b,$$

$$\frac{\hat{b}_n \big(\sum_{k=1}^n X_k - e^{-\hat{b}_n} \sum_{k=1}^n X_{k-1}\big)}{n(1 - e^{-\hat{b}_n})} =: \hat{a}_n \xrightarrow{\text{p}} a.$$

- The decreasing speed of $(\hat{b}_n - b, \hat{a}_n - a)$ as $n \to \infty$.

L–Ma ('15): For $1 < \alpha < (1+\sqrt{5})/2$, as $n \to \infty$,

$$n^{(\alpha-1)/\alpha^2}(\hat{b}_n - b, \hat{a}_n - a) \xrightarrow{\mathrm{d}} -e^b(1, ab^{-1})S_1^{-1}S_2, \tag{4}$$

where (S_1, S_2) has characteristic function explicitly known.

Open Problem The asymptotics of $(\hat{b}_n, \hat{a}_n)$ for $(1+\sqrt{5})/2 \le \alpha < 2$?

Proof of the central limit theorem:

- Exponential ergodicity of the stable CIR-model;
- Limit theorems of some random point processes;
- Limit theorems of partial sums like

$$\sum_{k=1}^{n} \varepsilon_k, \quad \sum_{k=1}^{n} \frac{\varepsilon_k}{1+X_{k-1}}, \quad \sum_{k=1}^{n} X_{k-1}^2, \quad \sum_{k=1}^{n} X_{k-1}\varepsilon_k.$$

The talk: ergodicities for measure-valued processes including the above model.

2. Measure-valued branching processes

Let E be a Lusin topological space ($=$ Borel subset of a compact metric space). Let $M(E)$ be the set of finite Borel measures on E.

- A Markov process in $M(E)$ is called a MB-process if its transition semigroup $(Q_t)_{t\geq 0}$ satisfies the (regular) branching property:

$$\int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\mu, \mathrm{d}\nu) = \exp\{-\mu(V_t f)\}, \qquad f \in B(E)^+, \tag{5}$$

where $\nu(f) = \int_E f d\nu$ and

$$V_t f(x) = -\log \int_{M(E)} \mathrm{e}^{-\nu(f)} Q_t(\delta_x, \mathrm{d}\nu), \qquad x \in E. \tag{6}$$

The operators $(V_t)_{t\geq 0}$ satisfy $V_s V_t = V_{s+t}$, called the cumulant semigroup.

Example 1 ($E =$ singleton) $\qquad dX_t = \sqrt[\alpha]{\alpha c X_{t-}} dZ_t - b X_{t-} dt.$

3. Decomposability and immigration structures

Say a probability N on $M(E)$ is decomposable or C-excessive for $(Q_t)_{t\geq 0}$ if there are probabilities $(N_t)_{t\geq 0}$ such that, for $t \geq 0$,

$$(NQ_t) * N_t = N \qquad \text{(formally } NQ_t \preccurlyeq N \text{ by convolution).} \tag{7}$$

- In this case, the family $(N_t)_{t\geq 0}$ is a skew-convolution semigroup (SC-semigroup), i.e.,

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r, t \geq 0. \tag{8}$$

- A transition semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$ is defined by (immigration process):

$$Q_t^N(\mu, \cdot) = Q_t(\mu, \cdot) * N_t, \qquad t \geq 0, \mu \in M(E). \tag{9}$$

  - $N^\infty := \lim_{t\to\infty} NQ_t$ exists and is an equilibrium of $(Q_t)_{t\geq 0}$;
  - $N_\infty := \lim_{t\to\infty} N_t$ exists and is an equilibrium of $(Q_t^N)_{t\geq 0}$;
  - $N = N^\infty * N_\infty$. Write $N \in \mathscr{E}_p^*(Q)$ if $N = N_\infty$.

Results to be presented: (i) Representations of $N \in \mathscr{E}_p^*(Q)$; (ii) Exponential ergodicities.

4. Dawson–Watanabe superprocesses

Let $c \in B(E)^+$, $b \in B(E)$ and $\eta(x, \mathrm{d}y)$ be a bounded kernel on E. Let $H(x, \mathrm{d}\nu)$ be a kernel from E to $M(E)$ satisfying

$$\int_{M(E)} [\nu(1) \wedge \nu(1)^2 + \nu(\{x\}^c)] H(x, \mathrm{d}\nu) < \infty. \tag{10}$$

- The branching mechanism is an operator $\phi : B(E)^+ \to B(E)$ given by

$$\phi(x, f) = b(x)f(x) - \eta(x, f) + c(x)f(x)^2 + \int_{M(E)} [\mathrm{e}^{-\nu(f)} - 1 + f(x)\nu(\{x\})] H(x, \mathrm{d}\nu).$$

- The underlying process is a Borel right process ξ in E with semigroup $(P_t)_{t \geq 0}$.

The (Dawson–Watanabe) (ξ, ϕ)-superprocess is an MB-process in $M(E)$ with cumulant semigroup $(V_t)_{t \geq 0}$ defined by (Dynkin '94/'02; Li '11):

$$V_t f(x) = P_t f(x) - \int_0^t \mathrm{d}s \int_E \phi(y, V_s f) P_{t-s}(x, \mathrm{d}y). \tag{11}$$

Recall that $N \in \mathscr{E}_p^*(Q)$ iff N is a probability on $M(E)$ and there are probabilities $(N_t)_{t\geq 0}$ on $M(E)$ such that $\lim_{t\to\infty} N_t = N$ and

$$N = (NQ_t) * N_t, \qquad t \geq 0. \tag{12}$$

- Let $\mathscr{K}(P)$ be the set of entrance laws $\kappa = (\kappa_t)_{t>0}$ for $(P_t)_{t\geq 0}$, i.e.,

$$\kappa_r P_t = \kappa_{r+t}, \qquad r, t > 0. \tag{13}$$

- For $\kappa = (\kappa_t)_{t>0} \in \mathscr{K}(P)$ write

$$V_t(\kappa, f) = \lim_{r\to 0+} \kappa_r(V_{t-r} f), \quad t > 0, f \in B(E)^+. \tag{14}$$

- If $\kappa \in \mathscr{K}(P)$ is closable, i.e., $\kappa = (\kappa_0 P_t)_{t>0}$ for a measure κ_0, we have

$$V_t(\kappa, f) = \kappa_0(V_t f), \quad t > 0, f \in B(E)^+. \tag{15}$$

Theorem 1 (Lévy–Khintchine) *For a (ξ, ϕ)-superprocess, each $N \in \mathscr{E}_p^*(Q)$ with finite first moment has the representation, for $f \in B(E)^+$,*

$$L_N(f) = \exp\left\{ -\int_0^\infty \left[V_s(\kappa, f) + \int_{\mathscr{K}(P)} (1 - \mathrm{e}^{-V_s(\eta, f)}) F(\mathrm{d}\eta) \right] \mathrm{d}s \right\}, \qquad (16)$$

where $\kappa \in \mathscr{K}(P)$ and $F(\mathrm{d}\eta)$ is a σ-finite measure on $\mathscr{K}(P)$.

5. Ergodicities in total variation distance

Using another bounded kernel $\gamma(x, dy)$ on E, we can rewrite

$$\phi(x, f) = b(x)f(x) - \gamma(x, f) + c(x)f(x)^2 + \int_{M(E)} [e^{-\nu(f)} - 1 + \nu(f)]H(x, d\nu).$$

- The local projection of ϕ is the function ϕ_1 on $E \times [0, \infty)$ defined by:

$$\phi_1(x, z) = [b(x) - \gamma(x, 1)]z + c(x)z^2 + \int_{M^\circ} [e^{-z\nu(\{x\})} - 1 + z\nu(\{x\})]H(x, d\nu).$$

Condition A There is a “nice” function ϕ_* on $[0, \infty)$ such that $\inf_{x\in E} \phi_1(x, z) \geq \phi_*(z)$, $z \geq 0$ and $\int^\infty \phi_*(z)^{-1} dz < \infty$.

Proposition 2 *Under Condition A, for $t > 0$ the function $\bar{V}_t(x) := \lim_{\lambda\to\infty} V_t\lambda(x)$ is bounded on E and (extinction probability):*

$$Q_t(\mu, \{0\}) = e^{-\mu(\bar{V}_t)}, \qquad \mu \in M(E). \tag{17}$$

Suppose that Condition A holds. Let $N \in \mathscr{E}_p^*(Q)$ be given by (16).

Theorem 3 *For $t > 0$ and $\mu \in M(E)$,*

$$\|Q_t^N(\mu,\cdot) - Q_t^N(\nu,\cdot)\|_{\mathrm{var}} \leq 2(1 - \mathrm{e}^{-|\mu-\nu|(\bar{V}_t)}) \leq 2|\mu - \nu|(\bar{V}_t). \tag{18}$$

Corollary 4 (Strong Feller property) *For $t > 0$ and $\mu, \nu \in M(E)$,*

$$|Q_t^N F(\mu) - Q_t^N F(\nu)| \leq 2\|\bar{V}_t\|\|F\|\|\mu - \nu\|_{\mathrm{var}}. \tag{19}$$

Theorem 5 *For $t > 0$,*

$$\|N_t - N\|_{\mathrm{var}} \leq 2\int_0^\infty \Big[\pi_s(\kappa, \bar{V}_t) + \int_{\mathscr{K}(P)} \pi_s(\eta, \bar{V}_t)F(\mathrm{d}\eta)\Big]. \tag{20}$$

Corollary 6 *For $t > 0$ and $\mu \in M(E)$,*

$$\|Q_t^N(\mu,\cdot) - N\|_{\mathrm{var}} \leq 2\mu(\bar{V}_t) + 2\int_0^\infty \Big[\pi_s(\kappa, \bar{V}_t) + \int_{\mathscr{K}(P)} \pi_s(\eta, \bar{V}_t)F(\mathrm{d}\eta)\Big]\mathrm{d}s.$$

Corollary 7 (Ergodicities) *We have:*

(i) $\lim_{t\to\infty} \|N_t - N\|_{\text{var}} = 0$.

(ii) *If* $\beta_* := \inf_{x\in E}[b(x) - \gamma(x,1)] > 0$*, there exists* $C \geq 0$ *so that*

$$\|Q_t^N(\mu,\cdot) - N\|_{\text{var}} \leq C(1+\mu(1))\mathrm{e}^{-\beta_* t}, \qquad t \geq 0. \tag{21}$$

Remark There are similar results for the Wasserstein distance.

Example 2 (Stannat '03a/'03b) Exponential ergodicity in total variation distance for closable $\kappa = (\kappa_0 P_t)_{t>0}$ and $F = 0$.

Example 3 (Friesen '19+) Exponential ergodicity in Wasserstein distance for closable κ and $\mathrm{supp}(F) \subset \{$closable entrance laws$\}$.

Question What is new with inclosable entrance laws?

6. The boundary immigration

The absorbing-barrier Brownian motion (aBm) ξ in $(0,\infty)$ has transition density

$$p_t(x,y) := g_t(x-y) - g_t(x+y), \qquad x,y,t>0, \tag{22}$$

where $g_t(\cdot) =$ density of $N(0,t)$.

- ◦ Take the branching mechanism $\phi(x,f) = cf(x)^2$ ($c \geq 0$, $x > 0$).
- ◦ Let the entrance law $\kappa = (\kappa_t)_{t>0} \in \mathscr{K}(P)$ given by $\kappa_t(f) = \frac{\partial}{\partial x} P_t f(0+)$.

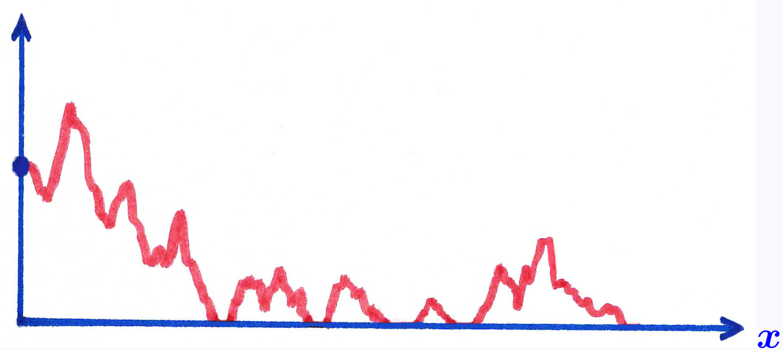
- From (ξ, ϕ, κ) a super aBm with immigration $\{Y_t : t \geq 0\}$ can be constructed.
- We have $Y_t(dx) = Z_t(x)dx$ and ($\partial_0 = \lim_{\varepsilon\to 0+} \epsilon^{-1}\delta_\varepsilon$)

$$\frac{\partial}{\partial t} Z_t(x) = \sqrt{2cZ_t(x)}\dot{W}_t(x) + \frac{1}{2}\Delta Z_t(x) - bZ_t(x) + a\partial_0 dt + \partial_0 dS_t. \tag{23}$$

where (S_t) is a pure-jump increasing Lévy process.

Continuous immigration of the super aBm with immigration: $Y_t(dx) = Z_t(x)dx$ and

$$\frac{\partial}{\partial t} Z_t(x) = \sqrt{2cZ_t(x)}\dot{W}_t(x) + \frac{1}{2}\Delta Z_t(x) - bZ_t(x) + a\partial_0 dt.$$

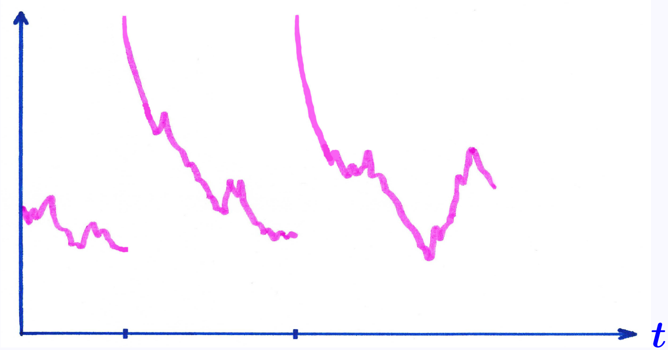


The density $Z_t = Z_t(x)$

$Z_t(0+) = 2a$: strong and continuing immigration pressure from the border

Discontinuous immigration of the super aBm with immigration: $Y_t(dx) = Z_t(x)dx$ and

$$\frac{\partial}{\partial t} Z_t(x) = \sqrt{2cZ_t(x)}\dot{W}_t(x) + \frac{1}{2}\Delta Z_t(x) - bZ_t(x) + \partial_0 dS_t.$$



The total mass process $t \mapsto Y_t(1)$

$Y_{\tau_i+}(1) = \infty$: infinite masses of the clusters of immigrants at the border

Two types of immigration at the border:

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thank you!